Chapter 3

Lie groups: basic definitions and general facts

In this chapter we will introduce the basic objects of Lie theory: Lie groups, their Lie algebras, the exponential map and the adjoint representation.

We will prove Cartan's theorem that a closed subgroup of a Lie group is a Lie group and discuss the correspondence between Lie (sub)-algebras and Lie (sub)-groups.

3.1 Lie groups and examples

We will assume working knowledge of the basics of differential geometry, but we will recall the basic definitions.

The reader who is not confident with the basics of differential geometry can consult for instance:

- F. Warner "Foundations of differentiable "manifolds and Lie groups"

— J. Lee. "Introduction to smooth manifolds".

Definition 3.1

A <u>Lie group</u> is a group G endowed with a structure of <u>smooth manifold</u> such that the multiplication $m: G \times G \longrightarrow G$ and inversion $i: G \longrightarrow G$ are <u>smooth maps</u>.

Recall

Definition 3.2

A <u>topological n-manifold</u> is a <u>second countable Hausdorff</u> space M such that every point in M has an <u>open neighborhood</u> that is <u>homeomorphic</u> to an <u>open subset of $\mathbb{R}^n$</u>.

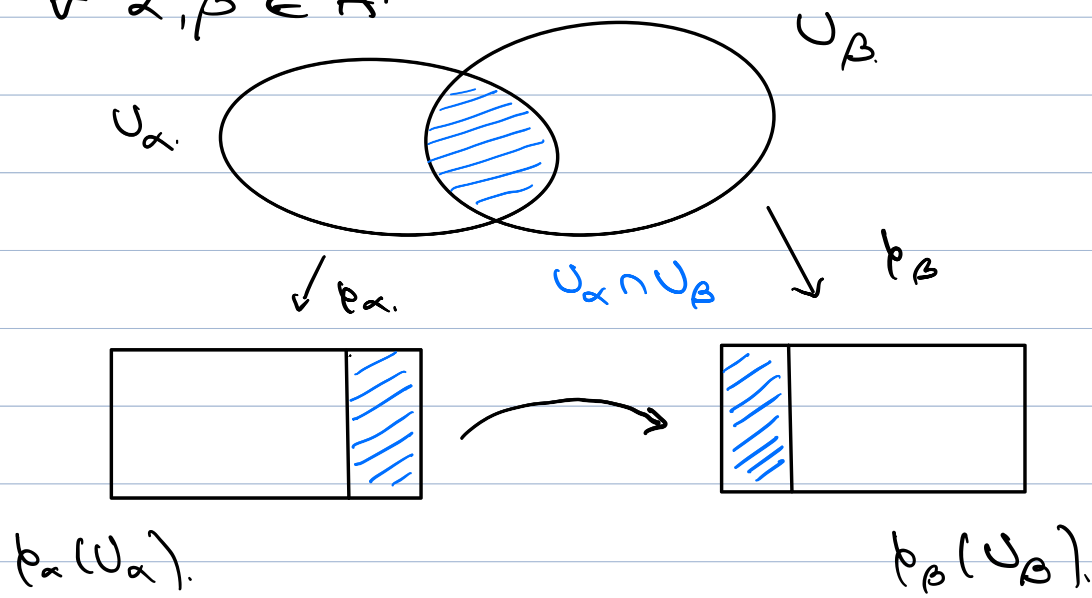
In this context a <u>chart</u> is a pair (U, φ) consisting of an open subset $U \subset M$ and a homeo $\varphi: U \longrightarrow \varphi(U) \subseteq \mathbb{R}^n$
$\hookrightarrow$ open.

Definition 3.3

A <u>smooth structure</u> on a top. n-manifold M is a <u>collection</u> $\mathcal{A} = \{ (U_\alpha, \varphi_\alpha), \alpha \in A \}$ <u>of charts</u> such that

1). $\bigcup_{\alpha \in A} U_\alpha = M$.

2). $\forall \alpha, \beta \in A$.



$$\varphi_\beta|_{U_\alpha \cap U_\beta} \circ \varphi_\alpha^{-1}|_{\varphi_\alpha(U_\alpha \cap U_\beta)} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

is a smooth map.

$= C^\infty$

3). $\mathcal{A}$ <u>is maximal</u> wrt the condition 2).

Any collection of charts satisfying 1) and 2) is called an <u>atlas</u> and one satisfying in addition 3) is called a <u>maximal atlas</u>. It is possible to prove that any atlas is contained in a maximal one.

See also: [Warner, 1.9 Lemma] for paracompactness.

Remark 3.4

A top. manifold is <u>paracompact</u> and has <u>countably many connected components</u>. See [Lee, Problem 1-5, pg. 30].

It is also helpful to recall what <u>a smooth map</u> is:
" C^∞

Let M be a smooth manifold of dimension n. and k be a nonnegative integer. Let $f: M \to \mathbb{R}^k$ be any function. We say that f <u>is smooth</u> if for every $p \in M$ there is a chart (U, φ) with $U \ni p$ such that $f \circ \varphi^{-1}$ <u>is smooth</u> on $\varphi(U) \subseteq \mathbb{R}^n$.
" C^∞

The definition easily generalizes to maps between manifolds.

Let M, N be smooth manifolds, and $F: M \to N$ be any map. We say that F is a smooth map if for every $p \in M$ there exist smooth charts (U, φ) with $U \ni p$ and (V, ψ) with $F(p) \in V$ such that $F(U) \subseteq V$ and $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$.

this is now a map between Euclidean domains.

We can now go through the examples of top. groups discussed in Chapter 2, and see which ones can be turned into Lie groups.

Example 3.5 [Cf. with Example 2.3]

Any countable discrete group is a 0-dimensional Lie group.

Example 3.6 [Cf. with Examples 2.4, 2.5].

$(\mathbb{R}^n, +)$, $(\mathbb{R}^*, \cdot)$, $(\mathbb{C}^*, \cdot)$ are Lie groups.

$\boxed{\text{Example 3.7}}$ [Cf. with Example 2.6]

$GL(n,\mathbb{R})$ is an open subset of $M_{n,n}(\mathbb{R})$ and as such it is a smooth n^2-manifold. The matrix product

$$M_{n,n}(\mathbb{R}) \times M_{n,n}(\mathbb{R}) \longrightarrow M_{n,n}(\mathbb{R})$$

is smooth, since it is polynomial, and so is the inverse $GL(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$ since

$$(A^{-1})_{ij} = \frac{\det M_{ji}}{\det A}.$$

$\boxed{\text{Example 3.8}}$ [Cf. with Ex 6 Sheet 1]

In general $\text{Homeo}(X)$ is not locally compact, for instance if X is a top n-manifold for $n \geq 1$. In particular it cannot be a Lie group.

$\boxed{\text{Example 3.9}}$ [Cf. with Example 2.14]

If (X,d) is a proper metric space then $\text{Iso}(X)$ is a locally compact Hausdorff group which may or may not be a Lie group.

For instance: if $(X,d) = (\mathbb{R}^n, d_{eucl})$ then $\mathrm{Iso}(X)$ is a Lie group. More generally, if (X,d) is a Riemannian manifold then $\mathrm{Iso}(X)$ is a Lie group [Myers - Steenrod '39].

In order to analyze more examples we need some additional tools from diff. geom.

We start by discussing the notion of regular submanifold.

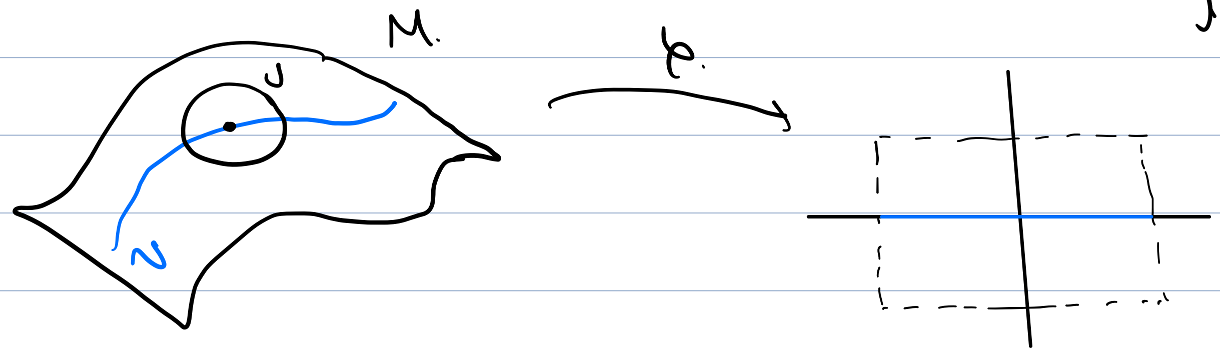
Let M be a smooth m-manifold.

$\boxed{\text{Definition 3.10}}$ [Regular submanifold]
A subspace $N \subset M$ is a regular n-submanifold if $\forall p \in N$ there is a chart (U, φ) at p (meaning $p \in U$) such that:

(1) $\phi(p) = 0$.

(2). $\phi(U) = (-1,1)^m$.

(3) $\phi(N \cap U) = \{x \in (-1,1)^m : x_{n+1} = \dots = x_m = 0.\}$



By restricting the charts from Def 3.10 to N we obtain a smooth n-manifold structure on N.

For us regular submanifolds will be relevant by the following.

$\boxed{\text{Theorem 3.11}}$

Let G be a Lie group and $H < G$ be a subgroup which is also a regular submanifold. Then H is a Lie group. (with the induced smooth structure).

The proof is left as an Exercise.

Exercise 3.12

- Find an example of a regular submanifold which is not a closed subset
- Show that if G is a Lie group and $H < G$ is a subgroup which is also a regular submanifold then H is closed in G.

A powerful tool to construct regular submanifolds is given by the following consequence of the implicit function theorem.

Theorem 3.13

Let $f : M \longrightarrow M'$ be a smooth map between smooth manifolds of dimensions respectively m and m'. Assume that f has constant rank k on M. Then $\forall q \in f(M)$ $f^{-1}(q)$ is a regular submanifold of M of dimension $m-k$.

Recall that the <u>rank</u> of f at $p \in M$ is the rank of the linear map.

$$D_p f : T_p M \longrightarrow T_{f(p)} M'$$

The notion of <u>tangent space</u> will be recalled below. We will be applying Thm 3.13 in the case where M is an open subset of $\mathbb{R}^N$ below, for some N.

Example 3.14

We show that $SL(n,\mathbb{R})$ and $O(n,\mathbb{R})$ are Lie groups.

(1). We claim that $SL(n,\mathbb{R})$ is a regular (n^2-1)-submanifold of $GL(n,\mathbb{R})$

It is sufficient to show that

$$\det : GL(n,\mathbb{R}) \longrightarrow \mathbb{R}^*$$

<u>has constant rank</u> equal to 1, as then Thm 3.13 applies.

We can compute

$$(D_A \det)(X) := \frac{d}{dt}\Big|_{t=0} \det(A+tX)$$

$$= \frac{d}{dt}\Big|_{t=0} \left[\det A \cdot \det(1 + tA^{-1}X)\right]$$

$$= (\det A)\,(D_I \det)\,(A^{-1}X).$$

Hence det has constant rank. The rank is 1 since (Exercise)

$$(D_I \det)(X) = \operatorname{tr} X.$$

(2) We claim that $O(n,\mathbb{R})$ is a regular $\frac{n(n-1)}{2}$ - submanifold of $GL(n,\mathbb{R})$.

It is sufficient to show that the map

$$f : GL(n,\mathbb{R}) \longrightarrow M_{n,n}(\mathbb{R})$$

$$A \longmapsto A \cdot {}^tA$$

has constant rank $= \frac{n(n+1)}{2}$ so the Thm 3.13 applies.

We compute

$$D_A f(X) := \frac{d}{dt}\Big|_{t=0} (A+tX)({}^tA + t\,{}^tX).$$

$$= \frac{d}{dt}\Big|_{t=0}\Big[{}^tAA + t\,({}^tXA + {}^tAX) + t^2\,{}^tXX \Big]$$

$$= {}^tXA + {}^tAX$$

Hence. $D_Af(X) = D_If({}^tXA)$ has constant rank.

The rank equals the dimension of $\{X + {}^tX : X \in M_{n\times n}(\mathbb{R})\}$ which is $\frac{n(n+1)}{2}$.

$\boxed{\text{Exercise 3.15}}$ [Cf. with Example 2.17]

Prove that $O(p,q)$ is a Lie group

3.2 Vector fields and Lie algebras.

In this section we recall a few facts about <u>smooth vector fields</u> and how they lead to an <u>algebraic object</u> called: <u>Lie algebra</u>.

Let M be a smooth manifold and $p \in M$

Recall that the ring of germs at p of smooth functions is

$$C^\infty(p) := \{ (U, f) : U \ni p \text{ is open}, \ f: U \to \mathbb{R} \text{ is smooth} \} / \sim$$

where $(U_1, f_1) \sim (U_2, f_2)$ if there is $p \in U_3 \subset U_1 \cap U_2$ open with $f_1|_{U_3} = f_2|_{U_3}$

This has a obvious ring structure. Note that $f(p)$ is well-defined if $f \in C^\infty(p)$.

$\boxed{\text{Definition 3.16}}$ [Tangent vector]

A tangent vector at p is a linear form $X_p : C^\infty(p) \to \mathbb{R}$ such that $\forall f, g \in C^\infty(p)$ it holds

$$X_p(fg) = f(p) X_p(g) + g(p) X_p(f).$$

$\longrightarrow$ Leibniz rule.

The set of tangent vectors at p forms a vector space denoted T_pM.

$\boxed{\text{Exercise 3.17}}$

If (U, φ) is any chart at p with $\varphi(p) = 0$

then: $\mathbb{R}^n \xrightarrow{\ \sim\ } T_pM$

$$v \longmapsto \left(f \longmapsto D_0 (f \circ \varphi^{-1})(v)\right)$$

is a vector space isomorphism.

The set of all tangent spaces can be organized into a space $TM = \bigsqcup_{p \in M} T_pM$, called the tangent bundle of M, with a natural smooth structure for which

$$\pi : TM \longrightarrow M$$
$$(v,p) \longmapsto p \quad , \ v \in T_pM$$

is a smooth map. [Lee, Proposition 3.18]

A smooth vector field is then a smooth section $M \longrightarrow TM$ of π.

We consider an alternative (equivalent) approach and introduce:

$\boxed{\text{Definition 3.18}}$ [Vector field]

A vector field on M is a map $X : M \longrightarrow TM$, $p \longmapsto X_p$ such that $X_p \in T_pM \ \forall p \in M$.

It is smooth if $\forall f \in C^\infty(M)$ the map $M \longrightarrow \mathbb{R}$

$$p \longmapsto X_p(f)$$

is smooth.

——— o ———

We want now to discuss the <u>local expression of a vector field</u> wrt a given chart.

So let (U,φ) be a chart on M. Denote $e_1, \dots, e_n$ the canonical basis of $\mathbb{R}^n$.

We get $\forall$ $1 \le i \le n$ a vector field $E^{(i)}$ on U defined by:

$$E^{(i)}_q(f) := D_{\varphi(q)}(f \circ \varphi^{-1})(e_i)$$

$q \in U$, $f \in C^\infty(U)$.

By Exercise 3.17 for every $q \in U$: $E^{(1)}_q, \dots, E^{(n)}_q$ is a basis of T_qM, hence if X is any vector field on U there are uniquely determined functions $g_1, \dots, g_n$ on U such that:

$$X_q = \sum_{i=1}^{n} g_i(q)\, E^{(i)}_q .$$

We have that X is smooth iff $g_1 \dots g_n$ are smooth. (Exercise)

We introduce a more abstract perspective on vector fields.

Recall that for a given field $\mathbb{K}$ a $\mathbb{K}$-algebra is a $\mathbb{K}$-vector space equipped with a bilinear product.

Definition 3.19 [Derivation of an algebra]
Let A be a $\mathbb{K}$-algebra where $\mathbb{K}$ is any field. A derivation of A is an endomorphism

$$\delta : A \longrightarrow A,$$

of the $\mathbb{K}$-vector space A such that

$$\delta(ab) = \delta(a) \cdot b + a \cdot \delta(b) \quad \forall a, b \in A.$$

We shall denote by $\mathrm{Der}(A)$ the space of derivations of A.

From now on $\mathrm{Vect}^{\infty}(M)$ (or simply $\mathrm{Vect}(M)$) will denote the space of smooth vector fields on M.

Proposition 3.20

The map $\alpha : \mathrm{Vect}^\infty(M) \longrightarrow \mathrm{End}(C^\infty(M))$ defined by $(\alpha X)(f)(p) := X_p(f)$ is an isomorphism onto its image $\mathrm{Der}(C^\infty(M))$.

Remark 3.21

Note that every $f \in C^\infty(M)$ defines an element of $C^\infty(p)$ $\forall p \in M$, namely the class of (M, f). Conversely $\forall U \ni p$ open and $f \in C^\infty(U)$ there is $F \in C^\infty(M)$ such that (M, F) and (U, f) are equivalent.

This can be checked by relying on the existence of cut-off functions $g \in C^\infty(M)$ s.t. $\mathrm{supp}\, g \subset U$ and $g \equiv 1$ on a neigh. of p.

Proof of Prop. 3.20

The fact that αX is a derivation follows from the Leibniz rule in the definition of tangent vector Def 3.16.

Conversely let $\delta : C^\infty(M) \longrightarrow C^\infty(M)$ be a derivation of $C^\infty(M)$.

Fix $p \in M$. Then: if $f_1, f_2 \in C^\infty(M)$ coincide in a neighborhood of p, it holds:
$$\delta(f_1)(p) = \delta(f_2)(p) \qquad (*)$$

Indeed, let $U \ni p$ open s.t. $f_1|_U = f_2|_U$.
Let $g \in C^\infty(M)$ such that:
$\operatorname{supp} g \subset U$ and $g \equiv 1$ in a neigh. of p.

$$\delta((f_1 - f_2)g)(p) \overset{\text{Leibniz}}{=} \delta(f_1 - f_2)(p)\, g(p) + (f_1 - f_2)(p)(\delta g)(p)$$
$$= \delta(f_1)(p) - \delta(f_2)(p).$$

$\delta((f_1-f_2)g)(p) = \delta(0)(p) = 0$

$f_1 = f_2$ on supp g.

Now define $X_p : C^\infty(p) \longrightarrow \mathbb{R}$ in the following way. Represent any (U, f) by an equivalent (M, F) using Remark 3.21 with $F \in C^\infty(M)$, and set:
$$X_p(f) := \delta F(p).$$

By (*) above, this X_p is well defined.
Using that δ is a derivation it is elementary to check that $X_p \in T_pM$ and X is a smooth vector field.

$\square$

We note that in general the composition of two derivations is not a derivation.
For instance if $\delta : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$

$$f \longmapsto f'$$

then $\delta^2 : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$,

$$f \longmapsto f''$$

is not a derivation. (Check this!!)

However, we have the following:

$\boxed{\text{Lemma 3.22}}$

Let $\delta_1, \delta_2 \in \mathrm{Der}(A)$. Then $\delta_1 \circ \delta_2 - \delta_2 \circ \delta_1 \in \mathrm{Der}(A)$.

Proof

$\delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$ clearly defines an endomorphism of A. We just need to verify the Leibniz rule.

To this aim we compute:

$$\delta_1\delta_2(a\cdot b) = \delta_1[\delta_2(a)b + a\,\delta_2(b)]$$
$$= \delta_1\delta_2(a)b + \underbrace{\delta_2(a)\,\delta_1(b)} + \underbrace{\delta_1(a)\delta_2(b)} + a\,\delta_1\delta_2(b)$$

$$\delta_2\delta_1(ab) = \delta_2[\delta_1(a)b + a\,\delta_1(b)]$$
$$= \delta_2\delta_1(a)b + \underbrace{\delta_1(a)\,\delta_2(b)} + \underbrace{\delta_2(a)\delta_1(b)} + a\cdot\delta_2\delta_1(b).$$

Hence.

$$(\delta_1\delta_2 - \delta_2\delta_1)(a\cdot b) = (\delta_1\delta_2 - \delta_2\delta_1)(a)\cdot b + a(\delta_1\delta_2 - \delta_2\delta_1)(b)$$

$\square$

We can apply this to $Vect^\infty(M)$: given $X, Y \in Vect^\infty(M)$ we conclude from Lemma 3.22 that $\alpha X\cdot\alpha Y - \alpha Y\cdot\alpha X \in Der(C^\infty(M))$ and hence by Prop. 3.20 it corresponds to an element of $Vect^\infty(M)$.

$\boxed{\text{Definition 3.23}}$ [Bracket of vector fields]
The <u>bracket</u> $[X,Y]$ of two <u>vector fields</u> $X, Y \in Vect^\infty(M)$ is the unique element in $Vect^\infty(M)$ such that

$$\alpha([X,Y]) = \alpha X\cdot\alpha Y - \alpha Y\cdot\alpha X.$$

More generally we can formalize this operation in the following:

Definition 3.24 [Bracket of endomorphisms]

If V is any $\mathbb{K}$-vector space, the bracket $[T_1, T_2] \in \mathrm{End}(V)$ of two endomorphisms T_1, T_2 is
$$[T_1, T_2] := T_1 T_2 - T_2 T_1$$

If A is a $\mathbb{K}$-algebra, the bracket operation is a bilinear map on $\mathrm{End}(A)$ preserving $\mathrm{Der}(A)$.

The map
$$\mathrm{End}(V) \times \mathrm{End}(V) \longrightarrow \mathrm{End}(V)$$
$$(T_1, T_2) \longmapsto [T_1, T_2]$$
satisfies:

1) It is bilinear;
2) (Antisymmetry) $[T_1, T_2] + [T_2, T_1] = 0$
3) (Jacobi) $[T_1, [T_2, T_3]] + [T_3, [T_1, T_2]] + [T_2, [T_3, T_1]] = 0.$

Remark 3.25

The Jacobi identity is a substitute of

associativity.

associativity.

Associativity would amount to

$[T_1,[T_2,T_3]] = [[T_1,T_2],T_3]$

Anti-symmetry $= -[T_3,[T_1,T_2]]$

Hence $[T_1,[T_2,T_3]] + [T_3,[T_1,T_2]] = 0.$

Definition 3.26 [Lie algebra]

A Lie algebra over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space $\mathfrak{g}$ endowed with a map $\mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$
$(x,y) \mapsto [x,y]$
satisfying the properties 1), 2) and 3) above.

Example 3.27

1) If V is a $\mathbb{K}$-vector space then $\mathrm{End}(V)$ endowed with the bracket is a Lie algebra.

2). If M is a smooth manifold, then $\mathrm{Vect}(M)$ endowed with the bracket is a Lie algebra.

3) $\mathbb{R}^3$ with the cross product is a Lie algebra.
~ or vector product

Definition 3.28 [Lie algebra homomorphism]

A $\mathbb{K}$-linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ of $\mathbb{K}$-Lie algebras is a Lie algebra homomorphism if $\phi([x,y]) = [\phi(x), \phi(y)]$ $\forall x, y \in \mathfrak{g}$.

o

Given a smooth map $\phi : M \to M'$ where M, M' are smooth manifolds in general there is no induced map $\mathrm{Vect}^\infty(M) \to \mathrm{Vect}^\infty(M')$

However there is such induced map if we assume that ϕ is a diffeomorphism.

See pag. 26 below for the definition of Derivative!

More generally we can introduce the following:

$\boxed{\text{Definition 3.29}}$ [ϕ-related vector fields]

We say that $X \in \mathrm{Vect}^\infty(M)$ and $X' \in \mathrm{Vect}^\infty(M')$ are ϕ-related if

$$X'_{\phi(m)} = D_m\phi\,(X_m) \quad \forall m \in M.$$

o

There is a useful algebraic reformulation:

Let $\phi^*(f) := f \circ \phi$, $f \in C^\infty(M')$

Then $\phi^* : C^\infty(M') \to C^\infty(M)$

is an algebra homomorphism.

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[dr, "f\circ\phi"'] & M' \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd}$$

Lemma 3.30

X and X' are ϕ-related iff the diagram

$$\begin{tikzcd} C^\infty(M') \arrow[r, "\phi^*"] \arrow[d, "\alpha X'"'] & C^\infty(M) \arrow[d, "\alpha X"] \\ C^\infty(M') \arrow[r, "\phi^*"] & C^\infty(M) \end{tikzcd}$$

commutes.

The proof is left as an Exercise.

Proposition 3.31

If X_i and X_i' are ϕ-related $i=1,2$, then $[X_1, X_2]$ and $[X_1', X_2']$ are ϕ-related.

Proof

By Lemma 3.30 above we have:

$$\phi^*\alpha([X_1', X_2']) = \phi^*(\alpha(X_1')\alpha(X_2') - \alpha(X_2')\alpha(X_1'))$$

Def. 3.23

$$= \alpha \cdot (X_1)\, \phi^* \alpha(X_2') - \alpha(X_2)\, \phi^* \alpha \cdot (X_1')$$ Lemma 3.30

$$= \alpha(X_1)\alpha(X_2)\phi^* - \alpha(X_2)\,\alpha(X_1)\,\phi^*$$ Lemma 3.30

$$= (\alpha(X_1)\,\alpha(X_2) - \alpha(X_2)\,\alpha(X_1))\, \phi^*$$

$$= \alpha \cdot ([X_1, X_2])\, \phi^*$$ Def 3.23

Hence $[X_1', X_2']$ and $[X_1, X_2]$ are ϕ-related by Lemma 3.30 again. converse implication! $\square$

We note that if $\phi : M \to M'$ is a diffeo then $\phi^* : C^\infty(M') \to C^\infty(M)$ is an isomorphism of algebras. What's the inverse?

Hence given $X \in \mathrm{Vect}^\infty(M)$ there is a unique $X' \in \mathrm{Vect}^\infty(M')$ which is ϕ-related to X, namely

$$\alpha X' = (\phi^*)^{-1} \alpha X \phi^* .$$

We will denote $X' := \phi_* X$.

Follows from Prop 3.31.

Corollary 3.32

If $\phi : M \to M'$ is a diffeomorphism then

$$\mathrm{Vect}^\infty(M) \to \mathrm{Vect}^\infty(M')$$

$$X \longmapsto \phi_* X.$$

is a <u>Lie algebra isomorphism</u>.

—— o ——

For the above definitions it is helpful to recall that: <u>the derivative, or tangent map</u>, or differential at $p \in M$ of a smooth map $\varphi : M \to M'$ is defined in the following way.

Let $X_p : C^\infty(p) \to \mathbb{R}$ be a <u>tangent vector</u> and $f \in C^\infty(\varphi(p))$, with a slight abuse of notation let (U, f) be a representative with $U \ni p$ open.

Then: $(D_p\varphi)(X_p)(f) := X_p(f \circ \varphi)$. check that it defines a tangent vector.

—— o ——

In the case when M is an <u>open subset</u> of a <u>finite dimensional vector space</u> over $\mathbb{R}$, V we will use some <u>conventions</u> and <u>identifications</u>.

Let $\Omega \subset V$ be open. We have the <u>identification</u> of the tangent space

$$\begin{aligned} V &\longrightarrow T_v\Omega \qquad v \in \Omega. \\ w &\longmapsto w_v \end{aligned}$$

$\leadsto$ $\quad w_v(f) = \left.\frac{d}{dt}\right|_{t=0} f(v+tw)$, $\quad f \in C^\infty(\Omega)$

If then $L : V \longrightarrow U$ is any linear map.

$$(D_v L)(w_v) = \left.\frac{d}{dt}\right|_{t=0} L(v+tw) = L(w)$$

In particular, $\forall \lambda \in V^*$

$$D_v \lambda (w_v) = \lambda(w) .$$

Idea : • the tangent space of a vector space is the vector space itself.

• the tangent map of a linear map is the linear map itself.

See [Lee, Proposition 3.13] for more details.

definition of $D_v L(w_v)$

$$(D_v L)(w_v)(f) = \left.\frac{d}{dt}\right|_{t=0} f(L(v+tw))$$

linearity $\leadsto$ $$= \left.\frac{d}{dt}\right|_{t=0} f(Lv + tLw)$$

$$= Lw_{Lv}(f) .$$